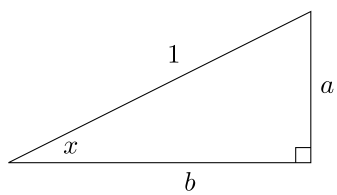
Calculus Summer Assignment
Due: First day of Class

Longleaf School of the Arts, 2023

This assignment has different identities that are very important to know in calculus and you're going to use most of them in derivatives, integrals, geometry, related rates, and others.

Basic Trig Identities.

I will now prove some basic trig identities, and then give you questions requiring that you use them to show that other trig identities are true. Let us consider a right triangle with angle x and hypotenuse of 1.



1. $\sin(x) = \frac{a}{1} = a$. Similarly, $\cos(x) = \frac{b}{1} = b$. And, $\tan(x) = \frac{a}{b}$. Substituting for a and b gives us our first basic trig identity. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.
2. Similarly, $\cot(x) = \frac{b}{a}$. Substituting again gives us $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$.
3. $\sec(x) = \frac{1}{b}$. Substituting for b gives us $\sec(x) = \dfrac{1}{\cos(x)}$.
4. Similarly, $\csc(x) = \frac{1}{a}$. Substituting for a gives us $\csc(x) = \dfrac{1}{\sin(x)}$.
5. Lastly, the Pythagorean Theorem tells us $a^2 + b^2 = 1^2$. Substituting for a and b gives us $\sin^2(x) + \cos^2(x) = 1$. (Note: It was decided long ago that $(\sin(x))^2$ was too awkward to write. So, $\sin^2(x)$ is the notation which means $(\sin(x))^2$. $\sin^2(x) = (\sin(x))^2$ and $\cos^2(x) = (\cos(x))^2$.)

Let us summarize what we have just shown. These are the five tools you will need to complete the problems below. These are the only five tools you have.

1. $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$.

2. $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$.

3. $\sec(x) = \dfrac{1}{\cos(x)}$.

4. $\csc(x) = \dfrac{1}{\sin(x)}$.

5. $\sin^2(x) + \cos^2(x) = 1$.

Now a couple of examples.

1. (Example 1): Show that $\dfrac{\tan(x)\csc(x)}{\sec(x)} = 1$.

$$\begin{aligned} \text{Substitute for the } \tan(x), \sec(x) \text{ and } \csc(x) \quad & \frac{\left(\frac{\sin(x)}{\cos(x)}\right)\left(\frac{1}{\sin(x)}\right)}{\frac{1}{\cos(x)}} \\ \text{Cancel } \sin(x) \quad & = \frac{\frac{1}{\cos(x)}}{\frac{1}{\cos(x)}} \\ \text{Cancel } \tfrac{1}{\cos(x)} \quad & = 1 \end{aligned}$$

I wrote what I did to help you see my thinking, and help you see how the tools above come into play. When you do these problems, there is no need to explain what you are doing. So, I expect to just see the following.

$$\begin{aligned} \frac{\tan(x)\csc(x)}{\sec(x)} & = \frac{\left(\frac{\sin(x)}{\cos(x)}\right)\left(\frac{1}{\sin(x)}\right)}{\frac{1}{\cos(x)}} \\ & = \frac{\frac{1}{\cos(x)}}{\frac{1}{\cos(x)}} \\ & = 1 \end{aligned}$$

Start on the left side, which will always be more complex, and work to achieve the right side. Since the right side is simpler, expect lots of things to cancel along the way.

2. (Example 2): Show that $\dfrac{\cos^2(x)}{1+\sin(x)} + \sin(x) = 1$ Often, there is more than one way to achieve the same end. I will demonstrate two different ways, with comments to help you see my thinking.

$$
\begin{aligned}
\text{Get a common denominator.} \quad & \frac{\cos^2(x)}{1+\sin(x)} + \frac{\sin(x)}{1}\left(\frac{1+\sin(x)}{1+\sin(x)}\right) \\
\text{Combine.} \quad & = \frac{\cos^2(x)+\sin(x)+\sin^2(x)}{1+\sin(x)} \\
\text{Use } \sin^2(x)+\cos^2(x)=1. \quad & = \frac{1+\sin(x)}{1+\sin(x)} \\
\text{Cancel.} \quad & = 1
\end{aligned}
$$

In showing these trig identities, there usually are not that many steps involved. If you find yourself taking 8 or more steps, stop and try a different approach. The difficulty of these problems lies in seeing what to do. Often, what to do in the next step is not apparent until you finish the step you are taking. Getting a common denominator (as we did above) is something you have done for almost half your life. But, seeing that we needed to use $\sin^2(x) + \cos^2(x) = 1$ was not apparent until we simplified the top.

Now, method #2. I will start by taking the trig identity $\sin^2(x)+\cos^2(x) = 1$ and subtracting $\sin^2(x)$ from both sides to form a new equation, $\cos^2(x) = 1 - \sin^2(x)$. (This is allowed. You can use algebra to transform the 5 equations above into different equivalent forms.)

$$
\begin{aligned}
\text{Substitue } \cos^2(x) = 1-\sin^2(x). \quad & \frac{1-\sin^2(x)}{1+\sin(x)} + \sin(x) \\
\text{Factor the top. Use the fact } A^2-B^2=(A-B)(A+B). \quad & \frac{(1-\sin(x))(1+\sin(x))}{1+\sin(x)} + \sin(x) \\
\text{Cancel } 1+\sin(x). \quad & 1-\sin(x)+\sin(x) \\
& = 1
\end{aligned}
$$

You are allowed to use any algebra tool you have ever learned. This will happen a lot in Calculus. Every technique you have ever learned in any math course needs to be right at your fingertips. You never know when some piece of knowledge will be useful. If you have not encountered $A^2 - B^2 = (A-B)(A+B)$, let me say a few words about it.

Multiply out $(A-B)(A+B) = A^2 + AB - BA + B^2 = A^2 - B^2$. So, I view $1-\sin^2(x)$ as $1^2 - \sin^2(x)$. ($A = 1$ and $B = \sin(x)$). That is why $1 - \sin^2(x) = (1-\sin(x))(1+\sin(x))$.

Last comment. Even though there are two approaches, both use the trig identity $\sin^2(x)+\cos^2(x) = 1$ at some point. This will be common in these problems. If you try different approaches, or if you work with others (which I encourage) on these problems, the "key" to showing each identity is true will be the same. In this case, $\sin^2(x) + \cos^2(x) = 1$ is the "key."

On to the problems.

1. Show that $\tan^2(x) + 1 = \sec^2(x)$. (Equivalently, show $\sec^2(x) - \tan^2(x) = 1$.)

2. Show that $\cot^2(x) + 1 = \csc^2(x)$. (Equivalently, show $\csc^2(x) - \cot^2(x) = 1$.)

3. Show that $-\dfrac{\tan(x)}{\cot(x)} + \sec^2(x) = 1$

4. Show $\dfrac{1 + \tan(x)}{\sec(x)} = \sin(x) + \cos(x)$

5. Show $\dfrac{\sin(x)}{1+\cos(x)} = \dfrac{1-\cos(x)}{\sin(x)}$. (Hint. Multiply fraction on left by $\dfrac{1-\cos(x)}{1-\cos(x)}$.)

6. Show $\dfrac{\sec(x)-1}{\sec(x)+1} + \dfrac{\cos(x)-1}{\cos(x)+1} = 0$

7. Show $\dfrac{1+\sin(x)}{1-\sin(x)} - \dfrac{1-\sin(x)}{1+\sin(x)} = 4\tan(x)\sec(x)$